Leray-α model and transition to turbulence in rough-wall boundary layers

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Abstract

In the present study, we use a fifth-order ordinary differential equation, a generalization of the Blasius equation derived from the Leray-α model, to examine the transition to turbulence in rough-wall boundary-layer flows. This equation, combined with a weaker formulation of the von Kármán log law modified to include the effects of surface roughness, provides a family of turbulent velocity profiles with two free parameters: the momentum thickness based Reynolds number and the roughness function. As a result, a skin-friction correlation is obtained and predictions of the transitional Reynolds numbers and maximal skin-friction values are made based on the roughness function.

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I. INTRODUCTION

Wall-bounded turbulent flows continue to be significant in both the natural environment and in engineering applications. As such, the turbulence community has persistently endeavored to describe the fluid dynamics in such flows [14]. In particular, the determination of skin-friction (or wall shear stress) has been of interest to many researchers.

For smooth-wall turbulent boundary layers, both direct and indirect techniques have been developed to determine skin-friction. These include the use of the momentum integral equation, correlations based on pressure measurements at the surface, and fitting mean velocity profiles based on a defect or power law [2].

However, in the rough-wall case [10, 11], direct measurement of skin-friction is often difficult. Indirect methods have included using friction velocity, u_τ, to estimate skin-friction. The modified Clauser method, for instance, has been used to approximate u_τ by fitting a logarithmic velocity profile to experimental data. However, as discussed by Acharya and Escudier [1], this technique is subject to large uncertainties because the degrees of freedom are increased from one (c_f) for a smooth surface to three (c_f, ϵ, $\Delta u/u_\tau$), where c_f is the skin-friction coefficient, ϵ is the error in origin, and $\Delta u/u_\tau$ is the roughness function. Alternative indirect techniques include determining u_τ using a velocity defect law, or power law formulations. Other correlations such as Bergstrom et al.'s skin-friction correlation with the ratio of the displacement and boundary-layer thicknesses have also been suggested [2].

In [7], a new theoretical method to derive a skin-friction correlation for smooth-wall turbulent boundary layers was presented. In this note we adopt it to rough-wall boundary-layer flows using the boundary-layer approximation of the Leray-α model of turbulence [4] and the von Kármán log law for rough walls [8, 9, 12]. The benefit of this approach is that it leads to a prediction of the critical Reynolds number based on momentum thickness R_θ^{crit}, the minimal Reynolds number where the transition to turbulence may occur. More precisely, we obtain the following dependence:

$$R_\theta^{\mathrm{crit}} = -51.8\Delta u/u_\tau + 365.5.$$

Since in a turbulent boundary layer, the skin-friction coefficients attains its maximum c_f^{max} at $R_\theta = R_\theta^{\mathrm{crit}}$, we also obtain the the following relation:

$$c_f^{\mathrm{max}} = 0.0063e^{0.1861\Delta u/u_\tau}.$$

II. BOUNDARY-LAYER APPROXIMATION OF THE LERAY-α MODEL

Proposed as a closure scheme for the Reynolds equations [4], the Leray-α model is written as

$$
\begin{cases}
\dfrac{\partial}{\partial t}\mathbf{v} + (\mathbf{u} \cdot \nabla)\mathbf{v} = \nu\Delta\mathbf{v} - \nabla p + f \\[2mm]
\nabla \cdot \mathbf{u} = 0 \\[2mm]
\mathbf{v} = \mathbf{u} - \alpha^2\Delta\mathbf{u},
\end{cases}
\tag{1}
$$

where $\mathbf{u}$ is the averaged physical velocity of the flow, p is the averaged pressure, f is a force, and $\nu > 0$ is the viscosity. The filter length scale α represents the averaged size of the Lagrangian fluctuations and is considered as a parameter of the flow. More specifically, we assume that α changes along the streamlines in the boundary layer, and is proportional to the thickness of the boundary layer (see [5]). Inspired by the Navier-Stokes-α model (see [3] and references therein), this model compared successfully with experimental data from turbulent channel, pipe, and boundary-layer flows for a wide range of Reynolds numbers.

We recall the boundary-layer approximation of the Leray-α model (see [7]). In the case of a zero-pressure gradient, consider a two-dimensional flow across a flat surface. Let x be the coordinate along the surface, y the coordinate normal to the surface, and $\mathbf{u} = (u, v)$ the velocity of the flow. Assuming that α is a function of x only, normalizing variables, and neglecting terms that are small near the boundary (see [6]), we arrive at a Prandtl-like boundary-layer approximation of the 2D Leray-α model:

$$
\begin{cases}
u\dfrac{\partial}{\partial x}w + v\dfrac{\partial}{\partial y}w = \dfrac{\partial^2}{\partial y^2}w \\[2mm]
v(x, y) = -\displaystyle\int_0^y \dfrac{\partial}{\partial x}u(x, z)\, dz \\[2mm]
w = u - \alpha^2\dfrac{\partial^2}{\partial y^2}u,
\end{cases}
\tag{2}
$$

where (u, v) are the components of the averaged velocity, p the averaged pressure, and $w = \left(1 - \alpha^2\frac{\partial^2}{\partial y^2}\right)u$.

The physical (non-slip) boundary conditions are $u(x, 0) = v(x, 0) = 0$, and $(u(x, y), v(x, y)) \to (1, 0)$ as $y \to \infty$. Looking for self-similar solutions to this system of the form

$$
u(x, y) = \frac{1}{\beta^2}h'(\xi/\beta), \qquad \alpha(x) = \beta\delta(x), \qquad \xi = \frac{y}{\delta(x)},
$$

with $\delta(x) = \sqrt{x}$, we reduce (2) to the following generalization of the celebrated Blasius equation:

$$
m''' + \frac{1}{2}hm'' = 0, \qquad m = h - h''.
\tag{3}
$$

3

The physical boundary conditions for (3) are $h(0) = h'(0) = 0$ and $h'(\xi) \to \beta^2$ as $\xi \to \infty$. This equation describes horizontal velocity profiles $\{h'(\cdot)\}$ in transitional and turbulent boundary layers with zero pressure gradients. In [6] it was proved that the above boundary value problem has a two parameter family of solutions with the parameters being

$$a := h''(0), \qquad b := h'''(0).$$

From the derivation of (3) it follows that the averaged horizontal velocity profiles u for a fixed horizontal coordinate x_0 is modeled by

$$u(x_0, y) = \frac{u_e}{\beta^2} h'\left(\frac{y}{\beta\sqrt{l_e l}}\right), \tag{4}$$

where y is the vertical coordinate, u_e is the horizontal velocity of the external flow, h is a solution to (3), $\beta = (\lim_{y\to\infty} h'(y))^{1/2}$, l is a local length scale that has to be determined, $R_l = u_e l/\nu$, and l_e is the external length scale $l_e = \nu/u_e$.

We now normalize quantities into wall coordinates

$$y^+ = \frac{u_\tau y}{\nu}, \qquad u^+ = \frac{u}{u_\tau}$$

where

$$u_\tau = \sqrt{\frac{1}{\rho}\tau} = \sqrt{\nu\frac{\partial u}{\partial y}\Big|_{y=0}},$$

and τ is the shear stress at the wall. Writing (4) in wall units, a three-parameter family of velocity profiles is obtained $u^+_{a,b,l}(\cdot)$:

$$u^+_{a,b,l}(y^+) = \frac{R_l^{1/4}}{\sqrt{a}\beta} h'\left(\frac{y^+\sqrt{\beta}}{R_l^{1/4}\sqrt{a}}\right). \tag{5}$$

III. SKIN-FRICTION CORRELATION

A critical step in deriving a skin-friction correlation is reducing a three-parameter family of the velocity profiles (5) to a two-parameter family of turbulent velocity profiles, which is achieved with a help of the von Kármán log law.

For smooth surfaces, the mean velocity profile for the inner region is commonly approximated with the von Kármán log law:

$$u^+ = \frac{1}{\kappa}\ln y^+ + B,$$

4

where the von Kármán constant, $\kappa \approx 0.4$, and $B \approx 5$, are empirically determined constants. In the rough-wall case, the effects of uniform roughness are confined to the inner region, and are accounted for by modifying the semi-logarithmic part of the mean velocity profile. More specifically, Clauser [8, 9] showed that the semi-logarithmic region is displaced downward by an amount $\Delta u / u_\tau$. This amount of downward shift is commonly referred to as the roughness function, and represents the velocity defect from the standard velocity distribution over a smooth wall, and indicates the additional wall shear stress due to the roughness. Accounting for the roughness effect, the log law can then be written as

$$u^+ = \frac{1}{\kappa} \ln \left[\frac{(y + \varepsilon) u_\tau}{\nu} \right] + B - \Delta B, \tag{6}$$

where ε is the shift at the origin for the rough wall, y is measured from the top of the roughness element, and $\Delta B = \Delta u / u_\tau$. The values of ε and ΔB are determined by matching experimental velocity profiles with (6).

We obtain turbulent velocity profiles by subjecting profiles (5) to three conditions of a weaker formulation of the von Kármán log law:

1. A turbulent velocity profile $u_t^+(y^+)$ has 3 inflection points in logarithmic coordinates.

2. The middle inflection point of $u_t^+(y^+)$ lies on the line

$$u^+ = \frac{1}{\kappa} \ln y^+ + B - \Delta B. \tag{7}$$

3. The line (7) is tangent to $u_t^+(y^+)$ at the middle inflection point.

Let us fix a Reynolds number based on momentum thickness

$$R_\theta = \int_0^\infty u^+ \left(1 - \frac{u^+}{u^+(\infty)} \right) dy^+. \tag{8}$$

Then (8) and conditions (ii) and (iii) determine all three parameters a, b, and l in (5). Therefore, conditions (ii) and (iii) reduce (5) down to a two-parameter family of turbulent profiles $\{ u_{R_\theta, \Delta B}^+ \}$. Fig. 1 shows the velocity profile for $R_\theta = 700$, $\Delta B = 1.57$ and corresponding experimental data of Osaka et al. [13] for d-type roughness.

Note that the skin-friction coefficient is uniquely determined by a velocity profile $c_f = 2 / u_{R_\theta, \Delta B}^+(\infty)^2$. Therefore, the skin-friction coefficient is now a function of R_θ and ΔB:
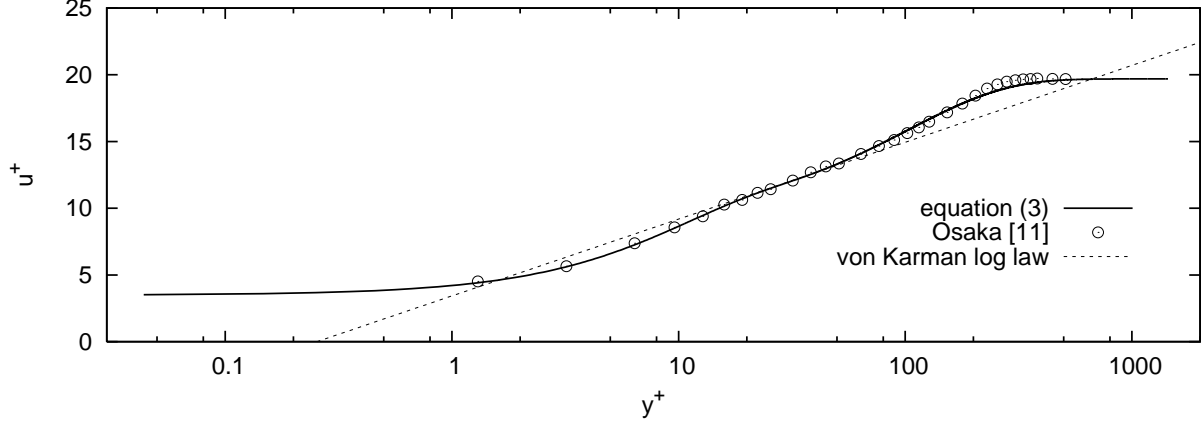
$$c_f = f(R_\theta, \Delta B). \tag{9}$$

FIG. 1: Velocity profile for $R_\theta = 700$, $\Delta B = 1.57$.
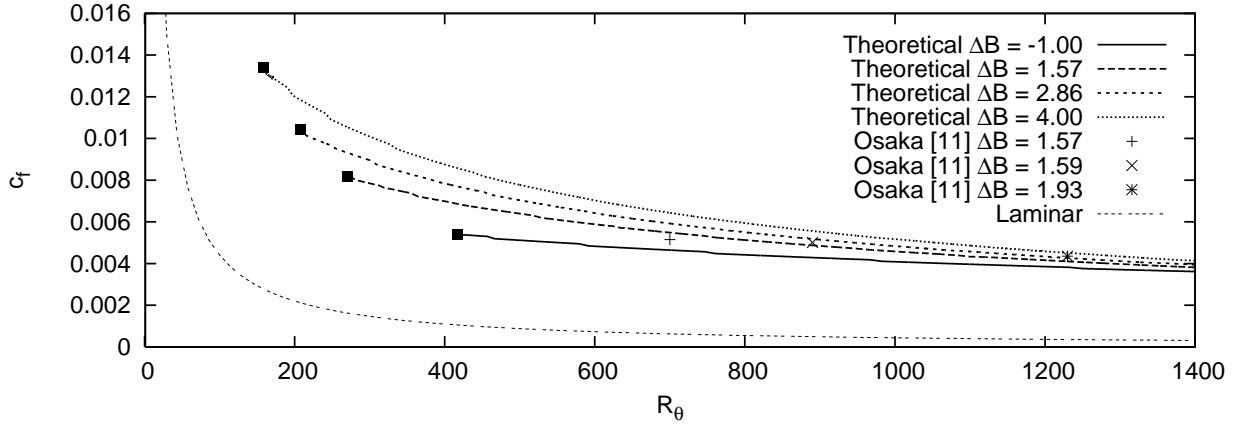

FIG. 2: Theoretical skin-friction correlation.

IV. THE ROUGHNESS FUNCTION AND TRANSITION TO TURBULENCE

The skin-friction law (9) is shown in Fig. 2 for several different values of roughness function, $\Delta B = \Delta u/u_\tau$. At the critical point where R_θ is at the minimum and c_f is at the maximal value, the second and third inflection points of the velocity profile collide, and the profile will then only have one inflection point for $R_\theta \leq R_\theta^{\mathrm{crit}}$. Therefore, the model indicates the minimal value of R_θ, i.e. R_θ^{crit}, for which a velocity profile can still be turbulent, i.e. the condition (i) is satisfied. One can interpret R_θ^{crit} as the minimal Reynolds number where the transition to turbulence may occur, often called the transitional Reynolds number in the literature.

Fig. 3 shows how this critical Reynolds number depends on the roughness function, which was obtained by computing R_θ^{crit} for sixteen different values of ΔB. The following linear correlation
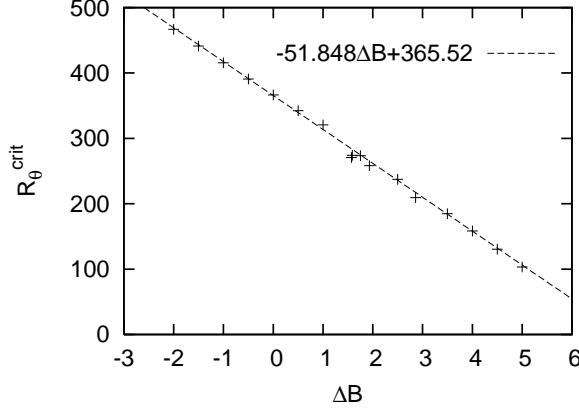
6

FIG. 3: Critical value of the Reynolds number.
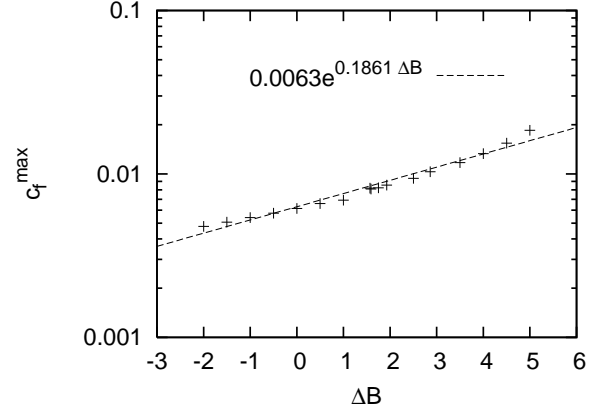

FIG. 4: Maximum of the skin-friction coefficient.

was found:

$$R_\theta^{\text{crit}} = -51.8\Delta B + 365.5. \tag{10}$$

Let us now denote $c_f^{\max} = f(R_\theta^{\text{crit}}, \Delta B)$. Note that $c_f^{\max}$ is the largest possible value of the skin-friction coefficient in the turbulent boundary layer. As demonstrated in Fig. 2, higher values of roughness function will allow velocity profiles to remain turbulent at lower Reynolds numbers, but furthermore will result in higher values of $c_f^{\max}$. Fig. 4, obtained by evaluating $c_f^{\max} = f(R_\theta^{\text{crit}}, \Delta B)$, shows how the roughness function is predicted to influence the maximal value of the skin-friction coefficient. The following correlation was found:

$$c_f^{\max} = 0.0063e^{0.1861\Delta B}.$$

V. CONCLUSION

Based on the Leray-α model of fluid turbulence, a generalized Blasius equation was formulated to describe streamwise velocity profiles in turbulent boundary layers with zero pressure gradients. Solutions of this fifth-order differential equation satisfying a weak formulation of the von Kármán log law form a two-parameter family. The two parameters are the Reynolds number based on momentum thickness R_θ and the roughness function ΔB. This leads to a skin friction correlation $c_f(R_\theta, \Delta B)$ and predictions of the critical Reynolds number $R_\theta^{\text{crot}}(\Delta B)$ and the maximal value of the skin-friction coefficient $c_f^{\max}(\Delta B)$. The critical Reynolds number is the minimal value of the transitional Reynolds number (which also depends on the intensity of the free-stream turbulence). In particular, it was shown that the greater ΔB is, the earlier the transition to turbulence may occur,

and furthermore the higher the skin-friction coefficient may peak.

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[1] M. Acharya, M. P. Escudier, Measurements of the wall shear stress in boundary-layer flows. *Fourth International Symposium on Turbulent Shear Flows* (1983), 277–286.

[2] D. J. Bergstrom, O. G. Marinade, M. F. Tachie, Skin Friction Correlation for Smooth and Rough Wall Turbulent Boundary Layers, *J. Fluids Engineering* **127** (2005), 1146–1153.

[3] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, S. Wynne, The Camassa-Holm equations and turbulence, *Physica D* **133** (1999), 49–65.

[4] A. Cheskidov, D. D. Holm, E. Olson, and E. S. Titi, On a Leray-α Model of Turbulence, *Proc. R. Soc. Lond. A* **461** (2005), 1–21.

[5] A. Cheskidov, Turbulent boundary layer equations, *C. R. Acad. Sci. Paris, Ser. I* **334** (2002), 423–427.

[6] A. Cheskidov, Boundary layer for the Navier-Stokes-alpha model of fluid turbulence, *Arch. Ration. Mech. Anal.* **172** (2004), 333-362.

[7] A. Cheskidov, Theoretical skin-friction law in a turbulent boundary layer, *Physics Letters A* **341** (2005), 487–494.

[8] F. H. Clauser, Turbulent boundary layers in adverse pressure gradients. *J. Aero. Sci.* **21** (1954), 91–108.

[9] F. H. Clauser, The turbulent boundary layer. *Adv. Appl. Mech.* **4** (1956), 1–51.

[10] N. Goldenfeld, Roughness-Induced critical phenomena in a turbulent flow, Phys. Rev. Lett. **96,** 044503 (2006).

[11] J. Jiménez, Turbulent flows over rough walls, *Annu. Rev. Fluid Mech.* **36** (2004), 173–96.

[12] J. Nikuradse, Laws of flow in rough pipes, NACA TM 1292, 1950. Translated from "Strmungsgesetze in rauhen Rohren" *Forsch. Arb. Ing.-Wes.* **361** (1933).

[13] H. Osaka and S. Mochizuki, Mean Flow Properties of a d-type rough wall boundary layer. *Proc. of the First KSME-JSME Thermal and Fluids Engineering Conference* **2** (1988), 88–93.

[14] H. Schlichting, *Boundary Layer Theory*, 7th ed., Mc Graw-Hill, 2000.